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Some existence results for quasi-linear elliptic problems via the fixed-point theorem of Tarski

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Abstract

In this paper, we study a family of quasilinear problems, related to Leray–Lions operators. We prove some existence theorems without any continuity condition on the right-hand side of the equation. This is possible by using the fixed-point theorem of Tarski.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geqslant 2)$. We consider the following problem:

$$\begin{aligned} &-\operatorname{div} A(x, \nabla u) + c_0 |u|^{p-2} u = F(x, u) \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

$p \in \,]1, \infty[$ and $c_0 \geqslant 0$.

This kind of problem appears in many fields: glaceology, climatology, reaction–diffusion problems, etc.

We consider here functions F, which are measurable and not Carathéodory. Under this weak assumption, the existence theorem of Drábek, Kufner and Nicolosi is not applicable. In the first part, the function: $t \to F(x,t)$ is increasing. We show that the framework of ordered spaces is convenient for problem (1), and permits us to give an existence theorem using the fixed-point theorem of Tarski. This was used first in [1].

In the second part, we show that the method can be used, without the monotony condition, but under a very weak condition on F. This fact was noticed in [5]. We can also consider degenerate problems.

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2. Functional framework

We give the assumptions on the nonlinear operator A and on the function F.

(i) $(x,\xi) \to A(x,\xi)$

$\Omega \times \mathbb{R}^N \to \mathbb{R}^N$

is a Carathéodory function, i.e. measurable with respect to x, continuous with respect to ξ_i ($i \in (1,\dots,N)$).

(ii) $(A(x,\xi) - A(x,\xi'), \xi - \xi') > 0 \quad \forall \xi,\ \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$ where $(\ ,\ )$ denotes the euclidean product in $\mathbb{R}^N$.

(iii) There exists a constant $q > 0$, such that

$$|A(x,\xi)| \geqslant q|\xi|^p, \quad \forall \xi \in \mathbb{R}^N$$

($|\xi|$ denotes the euclidean norm of ξ in $\mathbb{R}^N$).

(iv) There exists a constant $c_1 > 0$, and a function $b \in L^{p'}$ (where p' is the conjugate exponent of p), $b(x) \geqslant 0$ such that

$$|A(x,\xi)| \leqslant c_1(|\xi|^{p-1} + b(x)).$$

(v) $F : \begin{cases} (x,t) \to F(x,t) \\ \Omega \times \mathbb{R} \to \mathbb{R} \end{cases}$

is measurable (no continuity with respect to t is requested).

$(\text{vi})_1$ $t \to F(x,t)$

$\mathbb{R} \to \mathbb{R}$

is a nondecreasing function

$(\text{vi})_2$ There exists $K > 0$ such that

$$t \to F(x,t) + K|t|^{p-2}t$$

$$\mathbb{R} \to \mathbb{R}$$

is a nondecreasing function (for $1 < p < 2$, $|t|^{p-2}$ signifies: $\operatorname{sgn} t|t|^{p-1}$)

(vii) $F(x,t) \geqslant 0$ for $t \geqslant 0$ and $F(x,0) \not\equiv 0$.

(viii) There exist $a, c_2 \in L^q(\Omega)$, $a(x) \geqslant 0$, $c_2(x) \geqslant 0$, $q \geqslant p'$, $q > N//p$, $k > 0$ such that:

$$|Fx,t)| \leqslant a(x)|t|^k + c_2(x).$$

Definition 1. A solution u of (1) is an element of $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, such that

$$\int_\Omega (A(x,\nabla u), \nabla\varphi) + c_0 \int_\Omega |u|^{p-2}u\varphi = \int_\Omega F(x,u)\varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{2}$$

Definition 2. A subsolution (resp. supersolution) of (1) is an element $\underline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ (resp. $\overline{u}$), such that $\underline{u} \leqslant 0$ on $\partial\Omega$ (resp. $\overline{u} \geqslant 0$ on $\partial\Omega$) ($u \leqslant 0$ on $\partial\Omega$ means $u^+ \in W_0^{1,p}(\Omega)$) and: $\forall \varphi \in W_0^{1,p}(\Omega)$, $\varphi \geqslant 0$,

$$\int_\Omega (A(x, \nabla \underline{u}), \nabla \varphi) + c_0 \int_\Omega |\underline{u}|^{p-2} \underline{u} \varphi \leqslant \int_\Omega F(x, \underline{u}) \varphi \tag{3}$$

(resp.: $\forall \varphi \in W_0^{1,p}(\Omega)$, $\varphi \geqslant 0$,

$$\int_\Omega (A(x, \nabla \overline{u}), \nabla \varphi) + c_0 \int_\Omega |\overline{u}|^{p-2} \overline{u} \varphi \geqslant \int_\Omega F(x, \overline{u}) \varphi). \tag{4}$$

Let $v \in L^\infty(\Omega)$. Then by (viii) we have

$$|F(x, v(x))| \leqslant a(x) ||v||_\infty^k + c_2(x).$$

and $\tilde{F}_v : v \to F(x, v(x))$ is in $L^q(\Omega)$.

We consider the associated problem

$$\int_\Omega (A(x, \nabla u), \nabla \varphi) + c_0 \int_\Omega |u|^{p-2} u \varphi = \int_\Omega F(x, v(x)) \varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{5}$$

By the theorem of Leray–Lions [2] there exists a unique $u = Tv \in W_0^{1,p}(\Omega)$, solution of (5).

We have

Theorem 1. *Let $u \in W_0^{1,p}(\Omega)$ solution of*

$$\int_\Omega (A(x, \nabla u), \nabla \varphi) + c_0 \int_\Omega |u|^{p-2} u \varphi = \int_\Omega f \varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega), \tag{6}$$

with $f \geqslant 0$, $f \in L^q(\Omega)$, $q > N/p$, $q \geqslant p'$.

Then $u \geqslant 0$ and $u \in L^\infty(\Omega)$.

Proof. To prove that $u \geqslant 0$, we take $\varphi = u^-$ in (6), and get the result. To prove that u is in $L^\infty(\Omega)$, we adapt the proof of Proposition 3-1 in [3], with convenient modifications (cf. also [4]). $\square$

3. Existence theorem in the case of monotony

We assume here that F verifies assumption $(\mathrm{vi})_1$, i.e. the map: $t \to F(x, t)$ is a nondecreasing function ($\mathbb{R} \to \mathbb{R}$).

By Theorem 1, the operator $T : L^\infty(\Omega) \to L^\infty(\Omega)$ is well defined. In general, T is not continuous. With assumption $(\mathrm{vi})_1$, T is a monotone increasing operator, and we can use the theorem of Tarski (which does not request the continuity of T) to get a fixed point of T.

We define the natural order on $L^\infty(\Omega)$ by

$$u \leqslant v \quad \text{iff} \quad u(x) \leqslant v(x) \text{ a.e.} \quad \text{in } \Omega.$$

Now we study the operator T.

Proposition 1. *Under assumptions* (v), $(\mathrm{vi})_1$, (vii), (viii) *on F, $v_1 \geqslant v_2 \geqslant 0$ implies $Tv_1 \geqslant Tv_2$, or T is a monotone increasing operator on $L^\infty(\Omega)_+$ (the cone of positive elements of $L^\infty(\Omega)$).*

Proof. Let $u_1 = Tv_1$, $u_2 = Tv_2$.

As $v_1 \geqslant v_2 \geqslant 0$, $F(x, v_1(x)) \geqslant F(x, v_2(x)) \geqslant 0$, and u_1, u_2 are positive (first part of Theorem 1). We have: $\forall \varphi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega (A(x, \nabla u_i), \nabla \varphi) + c_0 \int_\Omega (u_i)^{p-1} \varphi = \int_\Omega F(x, v_i(x))\varphi, \quad i = (1,2).$$

Then $\forall \varphi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega (A(x, \nabla u_1) - A(x, \nabla u_2), \nabla \varphi) + c_0 \int_\Omega (u_1^{p-1} - u_2^{p-1})\varphi = \int_\Omega [F(x, v_1(x) - F(x, v_2(x))]\varphi.$$

We take $\varphi = (u_1 - u_2)^-$, and we get

$$u_1 \geqslant u_2.$$

In paragraph 2, we give a notion of sub(super)solution for problem (1). We have also a notion of sub(super)solution of the operator T. We show that both are the same in $L^\infty(\Omega)_+$. $\square$

Definition 3. A sub(super)solution for T is a $\underline{u}$ (resp. $\overline{u}$) in $L^\infty(\Omega)$ such that

$$\underline{u} \leqslant T\underline{u}$$

(resp. $\overline{u} \geqslant T\overline{u}$).

Lemma 1. *$\underline{u} = 0$ is a subsolution of (1) and a subsolution of T, but $\underline{u}$ is not a solution.*

Proof. $\underline{u} = 0$ is a subsolution of (1) (and not a solution) in the sense of Definition 2, by (vii). It is also a subsolution of T in the sense of Definition 3. Let $T(0) = v$; v is defined by

$$\int_\Omega (A(x, \nabla v), \nabla \varphi) + c_0 \int_\Omega |v|^{p-2} v\varphi = \int_\Omega F(x, 0)\varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

We have $v \geqslant 0$, as $F(x, 0) \geqslant 0$, by the first part of Theorem 1, then $T(0) = v \geqslant 0$, and $v \neq 0$; then $0 \leqslant T(0)$.

Lemma 2. *If $\overline{u} \geqslant 0$ is a supersolution of (1), then $\overline{u}$ is a supersolution of T, i.e. $T\overline{u} \leqslant \overline{u}$.*

Proof. Let $w = T\overline{u}$; w is defined by

$$\int_\Omega (A(x, \nabla w), \nabla \varphi) + c_0 \int_\Omega |w|^{p-2} w\varphi = \int_\Omega F(x, \overline{u})\varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

By assumption $\overline{u} \geqslant 0$; this implies $F(x, \overline{u}) \geqslant 0$ and then $w \geqslant 0$.

On the other hand, $\overline{u}$ is a supersolution in the sense of Definition 2:

$$\int_\Omega (A(x, \nabla \overline{u}), \nabla \varphi) + c_0 \int_\Omega \overline{u}^{\,p-1} \varphi \geqslant \int_\Omega F(x, \overline{u})\varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega),\ \varphi \geqslant 0.$$

then

$$\int_\Omega (A(x, \nabla \overline{u}) - A(x, \nabla w), \nabla \varphi)) + c_0 \int_\Omega (\overline{u}^{\,p-1} - w^{p-1})\varphi \geqslant 0, \quad \forall \varphi \in W_0^{1,p}(\Omega),\ \varphi \geqslant 0.$$

We take: $\varphi = (w - \overline{u})^+$, and we get

$$w \leqslant \overline{u}.$$

Then, with $\underline{u} = 0$

$$\underline{u} \leqslant T\underline{u} \leqslant T\overline{u} \leqslant \overline{u}. \tag{7}$$

We consider the interval order $[0, \overline{u}]$. We proved that T is monotone, increasing: $[0, \overline{u}] \to [0, \overline{u}]$. We know that $[0, \overline{u}]$ is a complete lattice in $L^\infty(\Omega)$. By the theorem of Tarski, there exists $u \in [0, \overline{u}]$, $u \neq 0$, such that $Tu = u$. But $u \in L^\infty(\Omega)$ implies $Tu \in W_0^{1,p}(\Omega)$ and the fixed point $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. It is a solution of (1). Then we proved:

Theorem 2. *If assumptions* (i)–(iv) *on* A, *and* (v), $(\mathrm{vi})_1$, (vii), (viii) *on* F *hold true, and furthermore if there exists a supersolution* $\overline{u} \geqslant 0$, *then there exists a solution* u *of problem* (1), *in* $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *such that* $0 \leqslant u \leqslant \overline{u}$, $u \neq 0$.

We give now an example where we can find a supersolution $\overline{u}$.

Proposition 2. *If* $c_0 > 0$ *and* $k < p - 1$, *there exists a constant* $M > 0$ *such that* $\overline{u} = M$ *is a supersolution of* (1).

Proof. If $\overline{u}$ is a supersolution, we have

$$c_0 \int_\Omega M^{p-1}\varphi \geqslant \int_\Omega F(x, M)\varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega),\ \varphi \geqslant 0.$$

A sufficient condition is

$$F(x, M) \leqslant c_0 M^{p-1}.$$

Using (viii), we find another sufficient condition

$$a(x)M^k + c_2(x) \leqslant c_0 M^{p-1}.$$

We can find M big enough, satisfying this inequality.

Remark 1. (1) In the case $p = 2$, the condition $k < p - 1$ means that F is sublinear.

(2) In Proposition 2, we cannot take $c_0 = 0$. In the case $c_0 = 0$ every constant is a subsolution, and we cannot find the supersolution M.

4. Existence theorem without monotony

Instead of assumption $(\mathrm{vi})_1$ on F, we assume $(\mathrm{vi})_2$. We write $G(x, t) = F(x, t) + K|t|^{p-2}t$. Problem (1) writes

$$\int_\Omega (A(x, \nabla u), \nabla\varphi) + (c_0 + K)\int_\Omega |u|^{p-2}u\varphi = \int_\Omega G(x, u)\varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{8}$$

Problem (8) is similar to (2), with $c_0 + K$ instead of c_0 and $G(x,u)$ instead of $F(x,u)$. Using Theorem 2, we get:

Theorem 3. *Under assumptions* (i)–(iv) *on* A, (v), $(\mathrm{vi})_2$, (vii), (viii) *on* F, *and furthermore we assume the existence of a supersolution* $\bar{u}$ *of* (9), *then there exists* $u \in W_0^{1,p} \cap L^\infty(\Omega)$, *such that* $0 \leqslant u \leqslant \bar{u}$, $u \neq 0$.

Proposition 2 holds true in the same way, because the additional term $K \int_\Omega |u|^{p-2} u \varphi$ disappears, and get two cases of existence of a supersolution.

Remark 2. We can treat the degenerate case in the same way, using the results of Drábek et al. [2].

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